# THE COINCIDENCE OF MEASURE ALGEBRAS UNDER AN EXCHANGEABLE PROBABILITY

BY

RICHARD A. OLSHEN

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DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

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# The Coincidence of Measure Algebras Under an Exchangeable Probability

By

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1. Introduction. This note is concerned with countably infinite product $\sigma$-fields and their invariant, tail, and exchangeable sub-$\sigma$-fields. Under an exchangeable probability the three sub-$\sigma$-fields coincide as measure algebras (the theorems (1) and (7)). An immediate consequence is the Hewitt-Savage 0-1 law. A later section includes examples which by and large preclude extensions of (1) and (7) to probabilities merely invariant under the shift. However, at least one interesting conjecture of David Freedman remains to be settled. I thank him for helpful conversations.

The results presented here serve to clarify and extend a remark by Halmos about power product probabilities ([4], p. 493). They also extend a theorem set forth by Meyer ([7], p. 150) to the effect that in a unilateral countable product space, under an exchangeable probability, exchangeable and tail $\sigma$-fields coincide as measure algebras.[1]

The final section contains the answer to a question posed in the paper [3] by Chung and Doob.

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1. Meyer attributes this result to Hewitt and Savage [4], and indeed one can argue that it is implicit there. I do not agree that it is "the main result of Hewitt and Savage".

2. Notation. If $(\Omega, \mathcal{B})$ is a measurable space and $I$ is either the set of integers $(Z)$ or the set of positive integers $(Z^+)$, then $\tilde{\Omega} = \tilde{\Omega}(I) = \Omega^I$, and $\tilde{\mathcal{B}}$ is the product σ-field. To avoid trivialities, assume that $\mathcal{B} \neq \{\Omega, \emptyset\}$. Here and in the remainder of the note $\pi$ refers to a fixed permutation of $I$ which leaves all but finitely many members fixed. To each $\pi$ corresponds a 1-1, bimeasurable map $a_\pi$ of $\tilde{\Omega}$ onto itself. More precisely, if $\tilde{\omega} \in \tilde{\Omega}$ has coordinates $\tilde{\omega}(i)$, then $(a_\pi \tilde{\omega})(i) = \tilde{\omega}(\pi(i))$. The set $E$ is exchangeable provided $E \in \tilde{\mathcal{B}}$ and $a_\pi E = E$ for each $\pi$. The collection of such sets is a σ-field, the exchangeable σ-field; it is denoted by $\tilde{\mathcal{E}}$. Context determines which $I$ is pertinent.

The shift $S$ maps $\tilde{\Omega}$ onto itself by $(S\tilde{\omega})(i) = \tilde{\omega}(i+1)$; plainly, $S$ is measurable. If $I = Z$, $S$ is 1-1 and bimeasurable, while if $I = Z^+$, it is decidedly not 1-1. And in the latter case if $(\Omega, \mathcal{B})$ is the Borel structure of a Borel set, then $S$ is bimeasurable iff $\Omega$ is countable. This is a special case of a difficult theorem of Purves [8]; however, it is rather easy to give a direct proof based on the fact that there exist Borel subsets of the unit square whose projections on an axis are not Borel. If $F \in \tilde{\mathcal{B}}$ and $S^{-1}F = F$, then $F$ is invariant. The invariant sets form a σ-field $\tilde{\mathcal{J}}$, the invariant σ-field. As with $\tilde{\mathcal{E}}$, the notation contains no reference to the index set $I$.

Suppose $J \subset I$. Define $\tilde{\mathcal{B}}(J)$ to be the σ-field of subsets $B$ of $\tilde{\mathcal{B}}$ with this special property: if $\tilde{\omega} \in B$ and $\tilde{\omega}'(j) = \tilde{\omega}(j)$ for all $j \in J$, then $\tilde{\omega}' \in B$. Of course $\tilde{\mathcal{B}}(I) = \tilde{\mathcal{B}}$.

3. The case $I = Z^+$. As the heading suggests, throughout this section $I$ is the set of positive integers. Define the future tail $\sigma$-field to be $\bigcap_{n=1}^{\infty} \tilde{\mathfrak{B}}(i: i \geq n)$, and denote it by $\tilde{\mathfrak{F}}^+$. It is well-known and easily proved that in the present case, $\tilde{\mathfrak{J}} \subset \tilde{\mathfrak{F}}^+ \subset \tilde{\mathfrak{E}}$, and the inclusions are proper. A probability $P$ on $\tilde{\mathfrak{B}}$ is an exchangeable probability provided $B \in \tilde{\mathfrak{B}}$ implies $P(a_\pi B) = P(B)$ for each $\pi$. A consequence of the following result is that under an exchangeable $P$, $\tilde{\mathfrak{J}}$ and $\tilde{\mathfrak{E}}$ are identical as measure algebras.

(1) Theorem. If $P$ is exchangeable and $E \in \tilde{\mathfrak{E}}$, then $P(E \triangle S^{-1}E) = 0$.

Proof. Fix $\gamma > 0$. A standard result from measure theory insures the existence of $m < \infty$ and $W \in \tilde{\mathfrak{B}}(i: 1 \leq i \leq m)$ for which

(2) $$P(W \triangle E) < \gamma \quad ,$$

where $\triangle$ means symmetric difference. Define $a_{\pi*}$ on $\tilde{\Omega}$ by

$$a_{\pi*}(\omega_1, \omega_2, \ldots \omega_m, \omega_{m+1}, \omega_{m+2}, \ldots) =$$

$$(\omega_{m+1}, \omega_1, \omega_2, \ldots \omega_m, \omega'_{m+2}, \ldots) \quad .$$

Then $a_{\pi*}W \triangle E = a_{\pi*}W \triangle a_{\pi*}E = a_{\pi*}(W \triangle E)$; consequently

(3) $$P(a_{\pi*}W \triangle E) = P(a_{\pi*}(W \triangle E)) = P(W \triangle E) \quad .$$

Now $S^{-1}W = a_{\pi*}W$. So according to (2) and (3),

(4) $$P(S^{-1}W \triangle W) = P(a_{\pi*}W \triangle W) \leq P(a_{\pi*}W \triangle E) + P(E \triangle W) < 2\gamma \quad .$$

Also,

(5) $$P(S^{-1}W \triangle S^{-1}E) = P(S^{-1}(W \triangle E)) = P(W \triangle E) \quad ,$$

for $S$ is measure-preserving. (It is enough that $P(S^{-1}C) = P(C)$ when $C$ is a cylinder, and this is given.)

Finally, (2), (4), and (5) together say that

$$P(E \triangle S^{-1}E) \leq P(E \triangle W) + P(W \triangle S^{-1}W) + P(S^{-1}W \triangle S^{-1}E) < 4\gamma \quad . \quad \diamond$$

(6) Corollary (Hewitt-Savage 0-1 law). If $P$ is a power product probability, then $E \in \tilde{\mathcal{E}}$ implies $P(E) = P^2(E)$.

Proof. Let $E^* = \lim\sup S^{-n}E$. In view of (1), $P(E \triangle E^*) = 0$. Now $E^* \in \tilde{\mathcal{J}} \subset \tilde{\mathfrak{F}}$, so apply the Kolmogorov 0-1 law. $\diamond$

4. The case $I = Z$. When $I = Z$, as is the case in this section, the relationships among invariant, tail, and exchangeable $\sigma$-fields are not so simple as when $I = Z^+$. In fact, there are several tail $\sigma$-fields of interest. Clearly, $\tilde{\mathfrak{F}}^+$ can be defined as in Section 3. But also the past tail $\sigma$-field $\tilde{\mathfrak{F}}_-$ merits study, where $\tilde{\mathfrak{F}}_- =_{df} \bigcap_{n=1}^{\infty} \tilde{\mathfrak{B}}(i\colon i \leq -n)$. Obviously, $\tilde{\mathfrak{F}}^+$ and $\tilde{\mathfrak{F}}_-$ are proper sub-$\sigma$-fields of $\tilde{\mathfrak{F}}_-^+$, the smallest $\sigma$-field containing them both. I learned from David Freedman (oral communication) that also $\tilde{\mathfrak{F}}_-^+$ is a proper sub-$\sigma$-field of $\tilde{\mathfrak{F}} =_{df} \bigcap_{n=1}^{\infty} \tilde{\mathfrak{B}}(i\colon |i| \geq n)$. He begins with the special case in which $\mathfrak{B}$ has four members as follows. Let $\underset{\sim}{\delta} = (\ldots\delta_{-1}, \delta_0, \delta_1, \ldots)$ be a (bilateral) sequence of random variables with $\delta_0, \delta_1, \ldots$ independent and identically distributed, and $P(\delta_0 = 1) = P(\delta_0 = -1) = \frac{1}{2}$.

Suppose that for $i \geq 1$, $\delta_{-i} = \delta_i(I_{[\delta_0=1]} - I_{[\delta_0=-1]})$. Then also $\delta_{-1}, \delta_{-2}, \ldots$ are independent; and, by the Kolmogorov 0-1 law, both past tail and future tail $\sigma$-fields of this process are trivial, but for every $n$, $\delta_0$ is determined by any pair $\{\delta_i, \delta_{-i}\}$ with $i \neq 0$, hence by $\{\delta_i : |i| > n\}$. Thus $\widetilde{\mathfrak{F}}$ is not trivial. To deduce the general case, pick $A \in \mathcal{B}$, $A \neq \Omega, \emptyset$. Fix $\omega_1 \in A$, $\omega_2 \in A^c$. For $B \in \mathcal{B}$, let $P(B) = 0$ if $\omega_1 \notin B$, $\omega_2 \notin B$; $P(B) = 1$ if $\omega_1 \in B$, $\omega_2 \in B$; $P(B) = \frac{1}{2}$ otherwise. By a variant of the Kolmogorov 0-1 law, $\widetilde{\mathfrak{F}}^+$ is trivial under the power product probability $\widetilde{P}$ on $\mathcal{B}(i: i \geq 0)$. Now for $n \in Z$, let $\xi_n(\omega) = 1$ if $\omega(n) \in A$, $\xi_n(\omega) = -1$ otherwise. Then $\{\xi_n\}_{n \geq 0} \sim \{\delta_n\}_{n \geq 0}$, where $X \sim Y$ means that the random objects $X$ and $Y$ have the same distribution. It is a very special case of the Tulcea extension theorem that $\widetilde{P}$ can be extended to $\widetilde{\mathcal{B}}$ in such a way that $\{\xi_n\}_{n \in Z} \sim \{\delta_n\}_{n \in Z}$ and $\widetilde{\mathfrak{F}}_-$ is trivial. Thus $\widetilde{\mathfrak{F}}^+_-$ is trivial, while $\widetilde{\mathfrak{F}}$ contains sets of probability $\frac{1}{2}$. The argument that $\mathfrak{F}^+_- \neq \mathfrak{F}$ implies a result in [5]. That is, if $\mathcal{C}_n$, $n=1,2,\ldots$, and $\mathcal{D}$ are $\sigma$-fields of subsets of a fixed space, and under a fixed probability $\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n$ is trivial, it does not necessarily follow that $\bigcap_{n=1}^{\infty} (\mathcal{C}_n \vee \mathcal{D})$ and $\mathcal{C} \vee \mathcal{D}$ coincide as measure algebras. (Of course set-theoretic inclusion in one direction is clear.) For if equality held, from two applications it would follow that under $\widetilde{P}$, $\widetilde{\mathfrak{F}}^+_-$ and $\widetilde{\mathfrak{F}}$ coincide as measure algebras. Section 5 contains a strengthened version of the foregoing example.

$\widetilde{\mathfrak{F}}$ consists precisely of those measurable sets whose measurability does not depend on any finite number of coordinates. (This characterization applies to $\widetilde{\mathfrak{F}}^+$ in the context of Section 3.) Patently, $\widetilde{\mathfrak{F}} \subset \widetilde{\mathcal{E}}$ properly, and $\widetilde{\mathcal{J}} \not\supseteq \widetilde{\mathfrak{F}}^+, \widetilde{\mathfrak{F}}_-, \widetilde{\mathfrak{F}}, \widetilde{\mathcal{E}}$.

(7) Theorem. Suppose $P$ is exchangeable. Then $\tilde{\mathcal{J}}$, $\tilde{\mathfrak{F}}^{+}$, $\tilde{\mathfrak{F}}_{-}$, $\tilde{\mathfrak{F}}_{-}^{+}$, $\tilde{\mathfrak{F}}$, $\tilde{\mathfrak{E}}$, and any intersection of these not contained in $\tilde{\mathfrak{F}}^{+} \cap \tilde{\mathfrak{F}}_{-} = \{\tilde{\Omega}, \emptyset\}$ coincide as measure algebras.

Proof. The argument given for (1), only slightly altered, shows that $E \in \tilde{\mathfrak{E}}$ implies the existence of $H \in \tilde{\mathfrak{J}}$ satisfying $P(E \triangle H) = 0$. Now take $F \in \tilde{\mathcal{J}}$, and fix $\gamma > 0$. There exists $m < \infty$ and $G \in \mathfrak{B}(i\colon |i| \leq m)$ satisfying $P(F \triangle G) < \gamma$. So

$$P(S^{-m}(F \triangle G)) = P(F \triangle S^{-m} G) < \gamma \quad ,$$

and $S^{-m} G \in \mathfrak{B}(i\colon i \geq 0)$. Thus there exists $G' \in \mathfrak{B}(i\colon i \geq 0)$ for which $P(F \triangle G') = 0$. Let $G^{*} = \limsup S^{-n} G'$. Then $G^{*} \in \tilde{\mathfrak{F}}^{+}$ and $P(F \triangle G^{*}) = 0$. The rest is obvious. $\diamond$

This argument shows that if $P$ is merely invariant under the shift, that is, if $P(S^{-1}A) = P(A)$ for each $A \in \tilde{\mathfrak{B}}$, then as measure algebras $\tilde{\mathfrak{J}} \subset \tilde{\mathfrak{F}}^{+}$. Also, $\tilde{\mathfrak{J}} \subset \tilde{\mathfrak{F}}_{-}$. Rosenblatt noticed this in [10]. Krengel and Sucheston [6] have shown more.

5. Generalizations. Motivated by the last paragraph, one might hope that many of the conclusions of (1) and (7) still hold if $P$ is not exchangeable but only invariant under the shift. The following examples substantiate my previous assertion that most of the conclusions no longer hold. In the first place, $\tilde{\mathfrak{F}}$ need not coincide with $\tilde{\mathfrak{F}}_{-}^{+}$. Assume that for each $i \in Z$, $\underset{\sim}{\delta}_i = (\ldots, \delta_{i,-1}, \delta_{i,0}, \delta_{i,1}, \ldots)$ is a

sequence of random variables distributed as $\ldots \delta_{-1}, \delta_0, \delta_1, \ldots$ in the last section, and that $\ldots \underset{\sim}{\delta}_{-1}, \underset{\sim}{\delta}_0, \underset{\sim}{\delta}_1, \ldots$ are independent. Let $\underset{\sim}{y}_i = (\ldots y_{i,-1}, y_{i,0}, y_{i,1} \ldots)$ be a (bilateral) sequence of sequences, where $y_{i,j} = \delta_{i+j,j}$. It is easy to see that the $\underset{\sim}{y}$ process is stationary and that its future tail $\sigma$-field is trivial because it is the smallest $\sigma$-field containing the future tail of each $\underset{\sim}{\delta}$ sequence. The past tail is also trivial. But the observations $\{\underset{\sim}{y}_i : |i| \geq n\}$ determine every $\delta_{i,0}$ for every n, and so the $\sigma$-field corresponding to $\tilde{\mathfrak{F}}$ is as rich as the Borel unit interval. Freedman has put forward the attractive conjecture that if $\mathfrak{B}$ is finite, then $\tilde{\mathfrak{F}}^+$ and $\tilde{\mathfrak{F}}$ conincide as measure algebras. A very special case of this conjecture will be mentioned at the end.

One direction in which something of (7) can be salvaged is the coincidence of $\tilde{\mathfrak{F}}^+$ and $\tilde{\mathfrak{F}}_-$ in case $\mathfrak{B}$ is finite. To see this, apply Theorem 2 of [9] first to $\tilde{\mathfrak{B}}(i: i \leq 0)$ and $S^{-1}$, then to $\tilde{\mathfrak{B}}(i: i \geq 0)$ and S, and find that both $\tilde{\mathfrak{F}}^+$ and $\tilde{\mathfrak{F}}_-$ correspond to Pinsker's maximal partition — what Rohlin and Sinai term $\pi(T)$. Be warned that in general $\tilde{\mathfrak{F}}^+$ and $\tilde{\mathfrak{F}}_-$ can be very different. For suppose $\ldots x_{-1}, x_0, x_1, \ldots$ is any sequence of independent, identically distributed, nondegenerate random variables, and for $i \in Z$, $\underset{\sim}{r}_i = (\ldots, x_{i-2}, x_{i-1}, x_i)$. Then $\{\underset{\sim}{r}_i\}$ is stationary; clearly its past tail is trivial, and its future tail is full. Of course $\{\underset{\sim}{r}_i\}$, as well as the previous $\{\underset{\sim}{y}_i\}$, can be realized on the unit interval. And note that $\{\underset{\sim}{r}_i\}$ is Markov; the remaining examples share this property.

It is hopeless to expect $\tilde{\mathfrak{J}}$ and $\tilde{\mathfrak{F}}^+$, or $\tilde{\mathfrak{F}}_-$, to coincide. A stationary process with trivial future or trivial past is mixing (see [1], page 121, or [2], Theorem 2), and there are stationary Markov processes which are ergodic but not mixing. In fact, any recurrent, countable state Markov chain with stationary transition probabilities and cyclically moving subclasses has a nontrivial future tail $\sigma$-field. Much more is shown in [2]. The third example of that paper is a stationary Markov chain with three states, and tail and exchangeable $\sigma$-fields which do not coincide as measure algebras. While the paper treats unilateral processes, the conclusions persit in the bilateral case. More precisely, Theorem 1 becomes: if $\{x_n\}_{n\in Z}$ is a stationary Markov chain with countable state space, and $A$ is determined measurably by the $x_n$'s but does not depend on any finite number of them, then $P(A|x_0 = i) = P^2(A|x_0 = i)$ for each state $i$. According to the extension of Corollary 1, $P(A|x_0 = i) = P(A|x_0 = j)$ if $i$ and $j$ are in the same cyclically moving subclass. Together, these facts and the aforementioned Example 3 substantiate the assertion about $\tilde{\mathfrak{F}}$ and $\tilde{\mathcal{E}}$. The facts along with the results of [2] also serve to establish Freedman's conjecture in the special case that $P$ is the measure of a Markov process.

6. <u>Intersections and products of σ-fields</u>. The distribution of the random variables $\underset{\sim}{\delta}$ in Section 4 can be utilized to solve a problem previously posed ([3], p. 414). Assume that $\Omega$ is a set and that for each real $t$ $\mathfrak{F}_t$ is a σ-field of subsets of $\Omega$. The $\mathfrak{F}_t$'s are nondecreasing. Let $\mathcal{B}_t$ be the σ-field of subsets of the interval $(-\infty,t]$, and for real $s$ and $s'$ let $\mathcal{B}_s \times \mathfrak{F}_{s'}$ be the product σ-field on $(-\infty,s] \times \Omega$. In the reference cited it was noted that for each real $a$,

$$\bigcap_{\delta>0} (\mathcal{B}_{a+\delta} \times \mathfrak{F}_{a+\delta}) = \bigcap_{\delta > 0} (\mathcal{B}_a \times \mathfrak{F}_{a+\delta}) \quad .$$

However there remained the problem as to whether these two σ-fields coincide as sets with $\mathcal{B}_a \times \mathfrak{F}_{a+}$, where $\mathfrak{F}_{a+} = \bigcap_{\delta>0} \mathfrak{F}_{a+\delta}$. Meyer has answered the question affirmatively when the product fields are augmented by the null sets of a product measure on $\mathcal{G} \times \bigvee_t \mathfrak{F}_t$, where $\mathcal{G}$ is the Borel σ-field on $(-\infty,\infty)$. But the question as originally posed has a negative answer. For there exist a family of σ-fields $\mathfrak{F}_t$ and a probability on $\mathcal{B}_1 \times \bigvee_t \mathfrak{F}_t$ under which $\bigcap_{\delta > 0} (\mathcal{B}_1 \times \mathfrak{F}_{1+\delta})$ and $\mathcal{B}_1 \times \mathfrak{F}_{1+}$ do not coincide as measure algebras.

Let $\Omega' = \{-1,1\}$, and give $\Omega'$ the discrete σ-field. Let $Z' = Z^+ \cup \{0\}$, and let $\Omega = (\Omega')^{Z'}$. Give $\Omega$ the product σ-field; call it $\mathcal{B}$. The σ-fields $\mathcal{B}(i\colon i \geq n)$ are defined as in Section 2. For $i \in Z'$ and $\tilde{\omega} \in \Omega$ let $X_i(\tilde{\omega}) = \tilde{\omega}(i)$, the $i^{th}$ coordinate of $\tilde{\omega}$. There exists a probability $P$ on $\mathcal{B}$ under which $(X_0,X_1,\ldots)$ has the same distribution as $(\delta_0,\delta_1,\ldots)$ in Section 4. For $i \in Z^+$ let $X_{-i}$ be the $i^{th}$ Rademacher function on the unit interval, so if $y \in [0,1]$ $X_{-i}(y)$ is $1$ or $-1$ according as the integer $j$ for which $j-1/2^i \leq y < j/2^i$ is odd or even.

Extend the domain of definition of $X_{-i}$ to the interval $(-\infty,1]$ by $X_{-i}(y) = 0$ if $y < 0$.

In an abuse of notation the X's will be viewed as functions on $(-\infty,1] \times \Omega$: for $i \geq 0$ $X_i((y,\tilde{\omega})) = \tilde{\omega}(i)$, and for $i < 0$ $X_i((y,\tilde{\omega})) = X_i(y)$. It is easy to show that P can be extended to $\mathcal{B}_1 \times \mathcal{B}$ in such a way that the sequence $\ldots X_{-1}, X_0, X_1, \ldots$ has the same distribution as the aforementioned sequence $\ldots \delta_{-1}, \delta_0, \delta_1, \ldots$ .

Now the $\mathfrak{F}_t$'s will be defined. When $t \leq 0$ let $\mathfrak{F}_t = \{\Omega, \emptyset\}$; when $t \geq 2$ let $\mathfrak{F}_t = \mathcal{B}$; when $n = 1,2,\ldots$ and $1/n{+}1 < t \leq 1/n$, let $\mathfrak{F}_t = \mathcal{B}(i\colon i \geq n-1)$.

For each $\delta > 0$ the event $A = [X_0 = 1]$ differs by a P-null event from an event in $\mathcal{B}_1 \times \mathfrak{F}_{1+\delta}$. Therefore A differs by a P-null event from an event in $\bigcap_{\delta > 0} (\mathcal{B}_1 \times \mathfrak{F}_{1+\delta})$. Now suppose that there exists an event $B^*$ in $\mathcal{B}_1 \times \mathfrak{F}_{1+}$ for which $P(B^* \Delta A) = 0$. This assumption will lead to a contradiction which will complete the argument that $\mathcal{B}_1 \times \mathfrak{F}_{1+}$ and $\bigcap_{\delta > 0} (\mathcal{B}_1 \times \mathfrak{F}_{1+\delta})$ do not coincide as measure algebras.

To begin fix $\varepsilon$, $0 < \varepsilon < 1/2$. It follows from elementary arguments that there exists $n \in Z^+$, $B_1, \ldots, B_n \in \mathcal{B}_1$, and $F_1, \ldots, F_n \in \mathfrak{F}_{1+}$ satisfying $P(A \Delta [\bigcup_{i=1}^{n} (B_i \times F_i)]) < \varepsilon$. The Kolmogorov 0-1 law implies that $P((-\infty,1] \times F)$ is 0 or 1 for each $F \in \mathfrak{F}_{1+}$. So for each fixed i either $P(B_i \times F) = 0$ or $P((B_i \times \Omega) \setminus (B_i \times F_i)) = 0$. With no loss of generality, assume that $F_i = \Omega$ for each i. Moreover, for each i also $P((B_i \times \Omega) \setminus ((B_i \cap [0,1]) \times \Omega)) = 0$ because this event is a subset of $[X_{-1} = 0]$. Again with no loss of generality, assume that each $B_i$ is a Borel subset of the interval $[0,1]$. Thus $B = \bigcup_{i=1}^{n} (B_i \times F_i)$ is determined measurably by $\{X_{-i}\}_{i \in Z+}$. Clearly A and B are independent. Recall that $P(A \triangle B) < \frac{1}{2}$. Now $P(A \triangle B) = P(A) + P(B) - 2P(A)P(B)$. Since $P(A) = P(X_0 = 1) = \frac{1}{2}$, $\frac{1}{2} + P(B) - P(B) < \frac{1}{2}$, which is impossible.

Loosely speaking, the foregoing construction shows that, when applied to σ-fields, the operations countable intersection and product do not commute. It would be interesting to know whether finite intersection and product commute.

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13. ABSTRACT

This note is concerned with countably infinite product σ-fields and their invariant, tail, and exchangeable sub-σ-fields. Under an exchangeable probability the three sub-σ-fields coincide as measure algebras (the theorems (1) and (7)). An immediate consequence is the Hewitt-Savage 0-1 law. A later section includes examples which by and large preclude extensions of (1) and (7) to probabilities merely invariant under the shift. However, at least one interesting conjecture of David Freedman remains to be settled.

The results presented here serve to clarify and extend a remark by Halmos about power product probabilities. They also extend a theorem set forth by Meyer to the effect that in a unilateral countable product space, under an exchangeable probability, exchangeable and tail σ-fields coincide as measure algebras.

The final section contains the answer to a question posed in the paper by Chung and Doob.

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